Bruhat Cubes as Product Categories: A Categorical and HoTT Reframing of Dyadic Uniformity and AI-Assisted Mathematical Discovery

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This paper recasts a recent AI-assisted combinatorial breakthrough in a deliberately abstract language: category theory and homotopy type theory (HoTT). The original result showed that certain Bruhat intervals in the symmetric group contain enormous Boolean-lattice substructures (hypercubes) whose dimension scales on the order of n log n. Here we ask a different question: what is the most transportable form of that phenomenon? Instead of treating the hypercube as a surprising combinatorial object, we treat it as an instance of "product emergence" inside a typically entangled order: a large interval becomes equivalent to a product of 2-element chains, i.e., a d-fold product of bits. The categorical lens makes this immediate. A poset is a thin category; an interval is a full subcategory; a Boolean lattice is a finite product. Under this view, the theorem's core content is the existence of large product subcategories inside Bruhat categories, forced by multiscale (dyadic) uniformity. HoTT then provides a second translation: the interval is a subtype equipped with an order relation, and the cube is a product type (Fin 2)^d; the theorem becomes an equivalence of structured types. Along the way, we isolate what is truly dyadic, what is truly Bruhat-specific, and what is methodological: how AI search can be aimed at detecting latent product structure.

Keywords: Bruhat order, symmetric group, Coxeter groups, Boolean lattice, hypercube interval, thin category, product category, posets as categories, dyadic uniformity, low-discrepancy nets, quasi–Monte Carlo, discrepancy theory, Richardson varieties, Kazhdan–Lusztig theory, cluster algebras, homotopy type theory, cubical type theory, structured equivalence, program synthesis, evolutionary search, AI-assisted discovery, conjecture generation, product emergence, multiscale constraints.

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1. Introduction: From “Hypercube in Bruhat” to “Product Emergence”

The headline version of the underlying result is almost too visual to be believed: inside the Bruhat order on permutations, there exist intervals that are literally hypercubes, with dimension growing on the order of n log n. It is a perfect story for the present moment: a classical structure, a surprising extremal object, and an AI-assisted pipeline that surfaced the pattern before humans extracted the definition and proved the theorem. But once the initial surprise settles, a more durable question emerges: what is the most transportable way to state what happened?

This paper proposes that the most portable description is not “a gigantic cube exists,” but rather: a large region of a highly entangled order factorizes into a product of binary degrees of freedom. That factorization is what a Boolean lattice is. A cube is not merely a geometric metaphor; it is the canonical poset-theoretic expression of independent 0/1 choices. Seen this way, the theorem is an instance of product emergence: a large interval inside Bruhat order behaves like a d-fold product of the 2-element chain. That structural reading is what makes the result travel to other posets, other Coxeter types, and other multiscale constraint systems.

Category theory and homotopy type theory (HoTT) are not introduced here as decoration. They are introduced because they give an exact language for “factorization into independent bits,” and because they expose which parts of the phenomenon are specific to dyadic uniformity and type A Bruhat order, and which parts are universal patterns that one can aim for elsewhere—possibly with the help of engineered search.

1.1 Two readings of the same theorem: extremal combinatorics vs structural algebra

There are two legitimate ways to read the theorem, and they lead to different kinds of follow-on work.

The first reading is the extremal combinatorics reading. Under this lens, the theorem answers a sharp size question: how large can a Boolean interval be inside Bruhat order on S_n? Since S_n has n! elements, any embedded Boolean lattice B_d has 2^d elements, so d is bounded above by log2(n!). The construction shows that, for n = 2^m, there are Bruhat intervals achieving d proportional to n log2 n, essentially saturating the counting ceiling up to constant factors. In this reading, the cube is an extremal witness: it says “the poset can contain product-like structure almost as large as information-theoretically possible.” The natural continuations are quantitative: improved bounds, asymptotics, rarity, and classification of near-extremizers.

The second reading is the structural algebra reading. Under this lens, the startling fact is not merely the size but the *form*: an interval inside a notoriously interdependent order becomes a pure product of d rank-1 pieces. That is an algebraic phenomenon: a complicated object exhibits

a large internal region where constraints decouple. The Boolean lattice is the canonical combinatorial signature of decoupling.

This second reading changes the questions you ask. Instead of "how big can d be," you ask:

- What mechanism forces a decoupling into independent flips?
- What does it mean, abstractly, for a poset to contain a huge product subobject?
- How can one detect or engineer such product regions in other settings?

In other words, the structural reading aims to extract a reusable template: not merely a family of examples, but a principle for when entangled order can factorize.

This is where category theory enters naturally. A poset is a thin category; a Boolean lattice is a finite product; "independent flips" are commuting squares; and "product emergence" becomes a precise categorical claim: a full subcategory of the Bruhat category is equivalent to a product of d copies of the 2-element category. HoTT then supplies a parallel formulation in the language of types: the interval is a subtype of permutations, the cube is a product type (Fin 2)^d, and the theorem becomes an order-respecting equivalence. These translations are not rhetorical. They are a way of making "factorization" the central object rather than "hypercube" as a spectacle.

1.2 Why abstraction now: portability across posets, Coxeter types, and scales

Abstraction is often criticized as removing the concrete content. Here, it does the opposite: it isolates the content that can survive transplant.

The original construction depends on two ingredients that are, in their native form, specialized:

- **Type A geometry:** permutations, permutation matrices, and the specific cover relations of Bruhat order in S_n.
- **Dyadic multiscale structure:** the n = 2^m regime, where binary partitions align perfectly across scales, making the "one point per dyadic rectangle" constraint exact.

If we keep the statement in its most concrete form—"here are explicit permutations in S_{2^m} forming a cube interval"—it is a powerful theorem but not obviously generalizable. Abstraction helps you separate "what is essential" from "what is contingent."

What is essential, structurally, is the emergence of a large product region: many local 0/1 degrees of freedom that commute and assemble into a Boolean lattice. What is contingent is the particular language used to describe those degrees of freedom (dyadic rectangles, permutation matrices) and the particular ambient poset (Bruhat in type A).

This is precisely the kind of situation where category theory earns its keep. By translating the phenomenon into "product subcategories" inside a thin category, you get a target that makes

sense anywhere: in any poset, in any Coxeter group's Bruhat order, and even in posets not directly tied to Coxeter theory. Similarly, by translating the dyadic constraints into a refinement system (a "site" of multiscale rectangles) and viewing DWD objects as global sections satisfying exact local compatibility, you get a template for changing scales:

- Replace dyadic by b-adic.
- Replace rectangles by other hierarchical partitions.
- Replace permutation matrices by other incidence objects.

HoTT contributes a different kind of portability. It lets you state "this interval is a cube" as an equivalence of structured types, emphasizing that the cube is a product type and the interval is a subtype with an order relation. This perspective naturally encourages questions like: what is the minimal data required to specify the equivalence? Can the construction be made computationally explicit as a function between types? Where does proof irrelevance matter, and where does structure matter?

So "why now" has two answers: the theorem has pushed the phenomenon to near-maximal scale (making structure impossible to ignore), and the AI-assisted discovery context invites us to formalize what the pipeline is actually detecting. Abstraction gives you a language to state that detection target: large latent product structure.

1.3 Reading guide: three threads (Bruhat, dyadic sites, categorical/HoTT translation)
The paper is organized around three threads that braid together but remain conceptually distinct.

The first thread is Bruhat. We keep only as much Bruhat theory as is needed to understand what "interval" and "cube" mean in this setting, and why Bruhat order is a natural place to measure entanglement versus factorization. This thread also keeps sight of what makes the result nontrivial: Bruhat order is not usually a product-like object, and large Boolean intervals are rare enough that their existence at near-maximal dimension reshapes the extremal landscape.

The second thread is dyadic sites. Here we treat dyadic uniformity not as an ad hoc combinatorial trick but as a multiscale compatibility condition. The dyadic rectangles are a refinement system; "one point per rectangle" is a gluing law across scales; and DWD permutations are global objects that satisfy exact local constraints at every resolution. This thread is where the QMC bridge lives conceptually, but the goal is broader: to treat "uniformity across scales" as a mechanism for producing modular degrees of freedom.

The third thread is categorical and HoTT translation. In category language, we recast posets as thin categories and cubes as products. In HoTT language, we recast intervals as types and cubes as product types, with the theorem becoming an equivalence that respects order. The point of

this thread is not to rewrite known math in new symbols; it is to make the structural claim crisp: the theorem exhibits an enormous product substructure, and the proof is the act of extracting a coordinate system (bits) that witnesses that product.

If you read with those three threads in mind, the narrative becomes clearer: the dyadic multiscale constraint creates local binary degrees of freedom; Bruhat order interprets those freedoms as commuting cover moves; category theory recognizes the result as an internal product region; HoTT recognizes it as a product-type equivalence. The theorem is the meeting point of all three, and the methodological moral is that AI-assisted search can be intentionally aimed at finding exactly such meeting points.

2. The Starting Point: Bruhat Intervals That Are Large Hypercubes

This paper begins from a concrete theorem in the most classical of settings: permutations ordered by Bruhat order. The theorem's punchline is easy to say and hard to internalize: for n = 2^m there exist Bruhat intervals whose internal poset structure is exactly a Boolean lattice of dimension proportional to n log n. In other words, inside a poset built from highly coupled constraints, one can find intervals that behave like a pure product of independent bits, and one can do so at nearly the largest dimension the ambient set size could possibly permit.

The purpose of this section is not to reproduce the proof, but to set the baseline vocabulary and identify the three essential ingredients we will later abstract: the ambient poset (Bruhat order), the structured subobject (a Boolean lattice, i.e., a hypercube), and the forcing mechanism (dyadic multiscale uniformity). Once those are on the table, it becomes natural to ask how much of the phenomenon is tied to the specific combinatorics of S_n and how much is a general "product emergence" principle that might reappear in other posets and other scales.

2.1 Minimal recap of the classical objects: S_n, Bruhat order, intervals [x,y]

The symmetric group S_n is the set of all permutations of n symbols. It is the canonical finite universe of "reorderings," and its size n! is already large enough that phenomena can be both enormous in absolute terms and rare in relative terms.

Bruhat order is a partial order on S_n that can be defined in several equivalent ways. For the purposes of this paper, the important facts are these:

- Bruhat order is graded: there is a rank function (often given by the inversion number, or length with respect to adjacent transpositions) so that moving upward corresponds to increasing length.

- The cover relations correspond to certain minimal “legal” increases in length. If you draw the Hasse diagram, an upward edge represents a smallest step upward in the partial order.
- The order is not at all product-like in general. It has many incomparable elements, many intersecting paths, and a strongly coupled combinatorial geometry. That coupling is why Bruhat order is central in representation theory and algebraic geometry: it encodes deep incidence relations, not merely a convenient ordering.

Given a poset, an interval is the natural unit of internal structure. For elements x <= y, the interval

[x,y] = { z in S_n : x <= z <= y }

is the sub-poset consisting of everything “between” x and y. Intervals are the right object to study for at least three reasons:

1. They are where local order-theoretic complexity lives. Many invariants and recursions depend only on intervals.
2. They are stable substructures: if you care about internal combinatorics, you care about what happens between endpoints.
3. They are a bridge to geometry. In Schubert and Richardson settings, intervals often encode incidence and singularity data.

So we start with a familiar picture: a very classical graded poset, and its naturally defined sub-posets.

2.2 The cube phenomenon: Boolean lattices inside Bruhat order at near-maximal dimension
A Boolean lattice B_d is the poset of all subsets of a d-element set ordered by inclusion. It is the prototypical poset of independent binary choices. It has:

- 2^d elements,
- ranks 0 through d with binomial layer sizes (d choose k),
- and cover relations given by adding one element to a subset.

This is why it is also called a d-dimensional hypercube: the Hasse diagram is the 1-skeleton of a cube in d dimensions, with each edge toggling one coordinate bit.

To say “a Bruhat interval is a hypercube” is therefore an exact statement: it says that, as a poset, the interval [x,y] is isomorphic to B_d. The proof must supply a concrete identification: a

bijection between elements of the interval and bitstrings of length d, such that Bruhat comparability matches subset inclusion and Bruhat covers match single-bit flips.

The remarkable part is the size of d. Since the interval lives inside S_n, it cannot have more than n! elements. Because B_d has 2^d elements, one gets the trivial ceiling

d <= log2(n!).

Using log2(n!) ≈ n log2 n − O(n), this ceiling is on the order of n log n. The theorem achieves d proportional to n log n, i.e., it reaches the ceiling up to constant factors. That is what "near-maximal" means here: the construction is not a minor improvement or a clever niche example. It is asymptotically pushing against the information limit imposed by the ambient set size.

This matters for two reasons. First, it resets intuition. If you believed large Boolean intervals should be scarce and modest in dimension, this shows that the poset has enough internal room for almost optimally large product-like regions. Second, it forces the question we care about in this paper: if such near-maximal product structure exists, what mechanism makes it possible, and can that mechanism be abstracted?

2.3 The dyadic mechanism in one sentence: multiscale uniformity forces independent local flips

The mechanism, stated as compactly as possible, is this:

A dyadic multiscale uniformity constraint on permutation matrices forces the permutation to decompose into many local two-choice blocks, and those choices commute, producing an interval isomorphic to a Boolean lattice.

This sentence compresses a lot, but it isolates the conceptual spine.

- "Permutation matrices" means we represent a permutation as n points in an n×n grid, one in each row and column.
- "Dyadic multiscale uniformity" means that, across a hierarchy of dyadic tilings of the grid, each dyadic rectangle of the prescribed shape contains exactly one point. This is the same kind of condition that appears in (0,m,2)-nets and quasi–Monte Carlo design.
- That condition is strong enough to force rigidity, but it is also hierarchical in a way that creates modularity. The grid breaks into complementary blocks at each scale, and in each block there are exactly two consistent local patterns.
- Each such block choice is a bit. Because blocks are arranged so their constraints do not interfere, the bits are independent: you can flip any subset of them and remain within the class.

Bruhat order then supplies the poset-theoretic interpretation: each bit flip corresponds to a cover move inside the interval, and the independence manifests as commuting squares. The interval becomes the poset of subsets of flips, hence a Boolean lattice.

Notice what this mechanism already suggests: dyadic structure is a sufficient forcing system, but perhaps not the only one. What we want is the abstract statement: a multiscale constraint system can force a large region to factorize into independent two-state modules.

2.4 Why we reframe: turning a construction into a reusable template

Why not simply celebrate the theorem as a striking construction and move on? Because the phenomenon it reveals is bigger than its native language.

If you keep the story purely in S_n terms, you are left with an impressive fact: there exist huge hypercube intervals when n is a power of two, built from dyadically well-distributed permutations. But the deeper opportunity is to turn this into a blueprint that can be reused in other contexts:

- In other Coxeter groups where Bruhat order exists but permutation matrices do not.
- In other posets where "interval structure" is central but the ambient objects are not permutations.
- In other scale systems where dyadic partitions are replaced by base-b partitions or by geometric decompositions driven by root systems or other hierarchies.

Reframing is what makes that possible. In categorical language, the key is not "a cube," but "a product subcategory" inside a thin category. A Boolean lattice is precisely a finite product of the 2-element chain, and "independent flips" are the data of a product decomposition. In HoTT language, the key is not "a set of permutations," but an equivalence between a subtype (the interval) and a product type (Fin 2)^d that respects the order relation.

Once stated that way, the theorem becomes a special case of a broader target: find conditions under which a complicated order contains a large region equivalent to a product of simple factors. The dyadic uniformity constraint is then reinterpreted as one example of a "multiscale gluing law" that yields modular degrees of freedom.

There is also a methodological reason to reframe. AI-assisted search is increasingly good at surfacing patterns, but patterns are not yet mathematics until they are compressed into definitions and proved. Category theory and HoTT are, at their best, compression machines: they identify the invariant core of a phenomenon and discard incidental coordinates. Reframing the theorem in those languages is therefore consistent with the paper's larger claim: AI can expand our exploratory reach, but humans still supply the act of conceptual compression that turns discovery into mathematics.

3. Posets as Thin Categories: The First Translation

The fastest route from “hypercube interval” to “product emergence” is to make one translation that category theory has always offered: treat a poset as a category with at most one morphism between any two objects. This move is so standard that it can be easy to underestimate its power. But in the present context, it does something very specific and very valuable: it replaces an eye-catching geometric noun (“hypercube”) with a structural verb (“product”). A Boolean lattice is not merely a pretty subposet; it is the canonical finite product of the two-element chain. So once we view posets as categories, “interval is a cube” becomes “interval is a product.” And “independent flips” become the categorical meaning of product coordinates.

This section lays out that translation carefully and uses it to restate the original theorem as a statement about large product subcategories inside the Bruhat category of S_n. Later sections will enrich this with dyadic “sites” and HoTT equivalences, but this first translation already captures the core phenomenon in the most minimal abstract language available.

3.1 From order to morphism: x <= y as the unique arrow x -> y

A poset (P, <=) can be regarded as a category C(P) defined as follows:

- The objects of C(P) are the elements of P.
- For objects x and y, there is a morphism x -> y if and only if x <= y.
- Because <= is antisymmetric and transitive, composition is forced: if x <= y and y <= z then x <= z, so the unique arrows compose to the unique arrow x -> z.
- There is at most one morphism between any two objects, because a poset has at most one truth value for x <= y. This is the defining feature of a thin category.

Seen this way, the axioms of a category are the axioms of a partial order:

- Identity arrows correspond to reflexivity: x <= x.
- Composition corresponds to transitivity: (x <= y and y <= z) implies x <= z.
- Antisymmetry corresponds to the “thinness plus equivalence” viewpoint: if x -> y and y -> x exist, then x and y must be the same object in the poset sense.

This translation is not merely formal. It forces a conceptual shift:

- In a poset, we ask “is x below y?”
- In the associated category, we ask “is there a morphism x -> y?”

That shift matters because category theory has a robust language for products, projections, and factorization—exactly the language we want when we say “a huge region behaves like independent bits.”

3.2 Intervals as full subcategories: Br[x,y] as a categorical object

Now apply this to Bruhat order. Let Br(S_n) denote the poset of permutations S_n with Bruhat order <=_B. Its associated thin category is what we might call the Bruhat category:

- Objects: permutations w in S_n.
- Morphism w -> v exists exactly when w <=_B v.
- Composition is forced by transitivity of Bruhat order.

An interval [x,y] in Bruhat order is:

[x,y] = { z in S_n : x <=_B z <=_B y }.

Categorically, this is not just a subset of objects; it is a canonical subcategory. Specifically, it is a full subcategory Br[x,y] of Br(S_n) with:

- Objects: all z between x and y.
- Morphisms: all morphisms among these objects that exist in Br(S_n), which in a thin category means: for any z1, z2 in [x,y], there is a morphism z1 -> z2 exactly when z1 <=_B z2.

“Full” matters even in a thin category. It means we do not artificially remove morphisms; the interval inherits all order relations among its elements. So the interval is an internal categorical object: a self-contained thin category whose structure is exactly the restricted Bruhat order.

This framing accomplishes a subtle but important simplification. If we want to compare intervals across different n, or across different Coxeter types, we can do so at the level of categories and functors:

- A map that preserves order becomes a functor between thin categories.
- A poset isomorphism becomes an equivalence of thin categories (indeed, an isomorphism).

So once we say “interval is a cube,” we are saying something very crisp: the thin category Br[x,y] is (isomorphic to) the thin category associated to the Boolean lattice.

3.3 Boolean lattices as products: B_d = (2)^d in Cat and in Poset

The Boolean lattice B_d can be presented in many equivalent ways. The standard one is:

- Objects: subsets of {1,…,d}.
- Order: inclusion.

But the categorical reading we want highlights a different presentation: B_d is the d-fold product of the two-element chain.

Let 2 denote the poset with two elements 0 < 1. As a thin category, 2 has two objects and one non-identity morphism 0 -> 1.

Then the product poset 2^d (Cartesian product of posets) has:

- Objects: d-tuples (b1,…,bd) with each bi in {0,1}.
- Order: componentwise. That is, (b1,…,bd) <= (c1,…,cd) if and only if bi <= ci for each i.
- This is isomorphic to the subset lattice: the tuple encodes which coordinates are 1, i.e., which elements are “included.”

Therefore:

B_d ≅ 2^d (as posets), hence also as thin categories.

This is the key conceptual payoff: a Boolean lattice is not merely “cube-shaped.” It is a finite product object in the category of posets (and likewise in the 2-category of thin categories).

Once you accept this, several features of cubes become categorical statements:

- The coordinate projections B_d -> 2 are the product projections.
- A “bit flip” corresponds to changing one coordinate while holding the others fixed.
- Commuting squares are forced by product structure: changing coordinate i and then j yields the same result as changing j and then i, because coordinates are independent.

So the slogan “hypercube = independent flips” becomes, in categorical terms, “product = independent coordinates.”

This is exactly why the categorical reframing is not cosmetic. It tells you what kind of structure you are really seeing: product structure, not merely a large subposet.

3.4 What the theorem becomes: existence of large product subcategories inside Br(S_n)
Now we can restate the core theorem in the language that this paper wants to emphasize.

Original combinatorial form (informal):
For n = 2^m, there exist x,y in S_n such that the Bruhat interval [x,y] is isomorphic to a Boolean lattice B_d with d proportional to n log n.

Categorical form:
For n = 2^m, the Bruhat thin category Br(S_n) contains full subcategories Br[x,y] that are isomorphic to the product category 2^d, with d proportional to n log n.

In other words, Br(S_n)—a category built from highly non-product global structure—contains huge internal regions that *are* product categories.

This is precisely the "product emergence" viewpoint. The cube is the visible manifestation of a deeper statement:

- Bruhat order is globally entangled.
- Yet certain dyadic constraints carve out an interval where morphisms factor coordinatewise.
- That interval is, categorically, a product of d rank-1 factors.

That restatement has immediate consequences for what kinds of questions become natural:

- How large can d be in other Coxeter types?
- What constraints (dyadic or otherwise) force the existence of large product subcategories?
- Are near-maximal product subcategories unique up to natural symmetries?
- Can we define a "product rank" of a poset (or of an interval) and search for intervals maximizing it?

Finally, it reframes the AI-assisted discovery story in a more operational way. A search pipeline can be designed to maximize signals of product structure: abundance of commuting squares, existence of many independent cover moves, or explicit coordinate projections. In categorical language, those are signals that a subcategory is approaching a product.

So the first translation is complete. It takes the theorem out of the register of surprising pictures and places it into the register of structural algebra: a near-maximal product object has been found inside a classical order. Subsequent sections will elaborate how dyadic multiscale constraints can be understood as a gluing principle that produces exactly these product

subcategories, and how HoTT offers a parallel expression of the same idea as an equivalence with a product type.

4. The Categorical Core: Product Subcategories and Factorization Signatures

Once we translate posets into thin categories, the "hypercube" result stops being primarily about a remarkable shape and becomes a statement about the appearance of a very specific categorical structure: a large product. In this section we isolate that structure and identify signatures that indicate it is present. The goal is not to restate definitions for their own sake, but to articulate what the theorem is really doing in a language that can travel.

The central claim of this paper is that the Bruhat-hypercube phenomenon is best understood as factorization: a region of the Bruhat category behaves as though it were built from many independent rank-1 components. That is what a product is: a universal way of packaging independent coordinates. And in a thin category, "independence" has an especially crisp meaning, because equalities of composites reduce to commuting diagrams of the simplest kind—squares.

This section therefore provides a structural reading of the result: what it means for a subcategory to be product-like, what local patterns certify that, how coordinates appear as functors, and why extreme product structure can coexist with extreme rarity.

4.1 What "product emergence" means categorically

In a general category, a product object A × B is characterized by a universal property: it comes with projection maps to A and B, and any object mapping to A and B factors uniquely through the product. In a thin category (a poset seen as a category), this universal property becomes extraordinarily concrete.

Suppose P is a poset and we consider it as a thin category. The product poset P1 × P2 is the cartesian product with componentwise order. Saying that a sub-poset Q of P is isomorphic to a product 2^d is therefore saying:

- Elements of Q can be labeled by d independent coordinates (bits).
- Comparability in Q is componentwise: one element is below another if and only if every coordinate is below.
- Covers in Q correspond to changing exactly one coordinate by one step (flipping one bit from 0 to 1, in the Boolean case).

This is already what "product emergence" means at the level of a statement: within an ambient poset P whose order is not componentwise, there exists a large interval Q whose order *is*

componentwise. But the word “emergence” is meant to point to something stronger than mere existence: it points to a mechanism by which a global tangle of constraints can locally decouple into modules.

Categorically, then, “product emergence” means:

- There is a full subcategory Q of the ambient thin category such that Q ≅ 2^d (or more generally ≅ C1 × ... × Ck for small factors Ci).
- The equivalence is not merely a bijection of objects; it is an identification of structure: it preserves and reflects the order relation, hence all morphisms.
- The effect is that any question internal to Q becomes coordinatewise. Paths, chains, joins/meets (where defined), and rank structure become the familiar behavior of a product.

In the Bruhat context, this is striking because Bruhat order is built from nontrivial interactions among reflections and reduced expressions. It is not “nearly a product.” Yet, under dyadic constraints, a large interval becomes literally a product of 2’s.

So the categorical core is not “a cube exists,” but “a maximal product region exists inside a category that is globally non-product.”

4.2 Commuting squares as product witnesses: why independence is a categorical property
In everyday language, the hypercube is built from “independent flips.” In categorical language, independence is witnessed by **commuting squares**.

In a product poset 2^d, there is a fundamental local property: if you have two different coordinates i and j that can be increased, then increasing i and then j yields the same result as increasing j and then i. In the Hasse diagram, this appears as a square:

u -> u_i
| |
v v
u_j -> u_{i,j}

where u_i means “u with coordinate i flipped,” and similarly u_{i,j} means both flipped.

In a thin category, a square commutes automatically if both composites exist, because there is at most one morphism from the lower-left object to the upper-right object. What is nontrivial is the existence pattern: do both paths exist, and do they land at the same object? In a product, the answer is systematically yes whenever two coordinates are available. In a generic poset, the answer is frequently no, because two local moves can interfere: taking one move may destroy the ability to take the other, or may lead to a different target.

This gives a robust “factorization signature”:

- Product-like behavior is detected by an abundance of commuting squares that close in the expected way.
- More strongly, a Boolean lattice is detected by a complete family of such squares arranged so that every subset of flips yields a unique object, independent of order.

The Bruhat-hypercube theorem can be read as the existence of a region where local Bruhat cover moves behave like independent coordinate increments. The dyadic constraints carve out exactly those moves that commute. Each move is local (a block choice), but the commutation is global: it asserts the absence of hidden interference among the modules.

This is why “independence” is properly categorical, not merely psychological. It is not that the moves “feel independent.” It is that the ambient structure admits a subregion where the two-step composites required by independence actually exist and coincide across the entire interval.

One can go further: in a Boolean lattice, you do not only have commuting squares. You have higher-dimensional commuting structure: any k distinct flips generate a k-dimensional face, and the endpoint obtained after all k flips is independent of the order in which they are applied. The presence of all these “higher commutations” is exactly what it means for the interval to be a product of k rank-1 factors. The square is simply the first nontrivial face you can draw.

So the categorical signature of product structure begins locally (squares), but its force is global (closure under all subsets of flips).

4.3 Rank, grading, and functorial coordinates: cube maps as product projections

A Boolean lattice is graded by Hamming weight: the rank of a bitstring is the number of 1’s. This grading is not incidental. It is the internal clock of the product: each coordinate flip increases rank by one, and rank simply counts how many coordinates have been turned on.

In Bruhat order, rank is given by length. In a cube interval, the length difference between the bottom element x and a point z in the interval is exactly the number of flips used to reach z. This alignment is part of why the cube structure is so rigid: the grading provides a consistency check that prevents “mystery shortcuts.”

Categorically, it is useful to view coordinates as **functors**. In a product 2^d, there are canonical projection functors:

pi_i : 2^d -> 2

that send a bitstring to its i-th coordinate. These projections are order-preserving maps (monotone maps), hence functors between thin categories.

Now consider a cube interval Q inside Bruhat order with an isomorphism:

phi : Q -> 2^d.

Then the coordinate maps on Q are:

ci = pi_i ∘ phi : Q -> 2.

Each ci is a functor that extracts the i-th bit. Taken together, the family {ci} is not merely descriptive; it can be viewed as the data that *defines* the product structure. If you can construct d monotone maps to 2 that behave like independent coordinates, you have essentially built the product decomposition.

This provides a powerful methodological bridge:

- If you want to detect product substructure, you can look for families of monotone "bit-valued invariants" that separate the interval's elements and satisfy the expected closure and commutation relations.
- If you want to engineer a search objective, you can reward candidates that admit many such coordinates—many stable binary invariants that behave independently under covers.

In the original theorem, the coordinates arise from block choices forced by dyadic uniformity. Each block contributes a 0/1 decision. The proof then shows that these decisions commute and that the Bruhat order relation is exactly componentwise inclusion on the chosen set of blocks. In categorical language, what the proof is really constructing is a family of coordinate functors that collectively exhibit an isomorphism to a product of 2's.

This is also where "interpretability after discovery" becomes concrete: the generator might output examples, but the human mathematical act is to discover the coordinate functors—the invariants that make the interval transparently a product.

4.4 Rigidity vs freedom: why product regions can be huge yet rare

At first glance, "huge and rare" sounds paradoxical. If a structure is huge, shouldn't it be easy to find? In a factorial universe like S_n, the paradox dissolves. "Huge" is absolute; "rare" is relative.

A Boolean lattice 2^d is rigid in structure: every element is uniquely determined by which subset of coordinates is on, and every comparability relation is forced by inclusion. There is almost no flexibility. Once you have fixed the endpoints x and y and fixed which flips are allowed, the entire interval is determined.

But that rigidity does not imply that such intervals are common. In fact, it suggests the opposite: to realize a giant product region inside a complicated poset, the ambient interactions must

cancel out almost perfectly across many degrees of freedom. Any small interference between two would-be independent moves breaks the square-closing property and destroys the product structure.

So why can the product region be huge? Because once the decoupling mechanism exists, it generates 2^d elements automatically. Independence exponentiates. One clean modular decomposition produces an exponential family of consistent objects.

And why can it be rare? Because the conditions required for modular decomposition are extremely stringent. A generic region of Bruhat order has abundant interference among local moves; the local geometry does not close into squares systematically, much less into higher faces. The dyadic uniformity constraint is special precisely because it is both strong and hierarchical: it forces exact balance at every scale while still allowing a two-choice pattern inside each complementary block. In other words, it supplies exactly the combination needed for huge freedom (many binary choices) under huge constraint (global multiscale consistency).

Categorically, we can summarize this as follows:

- Product structure is globally rigid: it fixes the entire subcategory once coordinates are chosen.
- Product structure, when present, creates enormous combinatorial volume: 2^d objects.
- The ambient category is generically non-product: product subcategories of large dimension are exceptional and require a forcing mechanism.

This “rigidity vs freedom” tension is not a defect; it is the signature of modularity. It is the same tension seen in other domains: strict interface constraints enable vast compositional freedom. Here, dyadic constraints act like an interface specification, and the flips are the compositional degrees of freedom.

This is the end of the categorical core: product emergence means the appearance of a large product subcategory; commuting squares and coordinate functors are signatures and witnesses of that product; and the coexistence of hugeness and rarity is explained by the exponential payoff of independence under stringent global compatibility. Subsequent sections will make the forcing mechanism itself more abstract by viewing dyadic uniformity as a gluing law on a multiscale refinement category, and will then recast the entire story in HoTT as an equivalence between an interval type and a product type.

5. Dyadic Uniformity as a Gluing Law: A Presheaf/Sheaf View

The dyadic condition that drives the hypercube construction is often presented as a combinatorial rule: place one permutation-matrix point in each dyadic rectangle at every scale. That phrasing is correct, but it leaves the mechanism feeling like a clever trick. The purpose of this section is to re-express the same condition as a **local-to-global compatibility law**—the kind of law that presheaves and sheaves were invented to formalize.

This shift matters because "one point per dyadic box" is not merely a global statistic. It is a family of exact constraints indexed by a refinement hierarchy. Each scale subdivides the grid; constraints at finer scales must be consistent with constraints at coarser scales; and the admissible permutations are precisely those global objects that satisfy all local constraints simultaneously. That is exactly the presheaf/sheaf posture: you specify what data looks like on pieces, specify restriction maps along refinements, and ask which global objects exist that glue perfectly.

When we adopt this viewpoint, the dyadic mechanism becomes less mysterious and more transportable. The hypercube does not appear because "binary is magic," but because the dyadic refinement system is a particularly clean multiscale site: it produces many local two-choice modules while maintaining strict compatibility across scales. Once you see it as a gluing law, it becomes natural to ask what happens for other bases, other partition hierarchies, and other ambient combinatorial objects.

5.1 The dyadic rectangle poset as a refinement category (the multiscale site)

Fix n = 2^m and consider the discrete square grid [n] × [n], with [n] = {0,1,…,n−1}. A dyadic interval in [n] is a block of consecutive integers whose length is a power of two and whose endpoints align with that power-of-two scale. Concretely, at scale k (where 0 <= k <= m), dyadic intervals have length 2^k and start at multiples of 2^k. A dyadic rectangle is then a Cartesian product of two dyadic intervals, one in the x-direction and one in the y-direction.

These rectangles naturally form a refinement hierarchy: a larger rectangle is subdivided into smaller rectangles by halving intervals. If we order dyadic rectangles by refinement (say, R' <= R if R' is contained in R and corresponds to a refinement piece), we obtain a poset of dyadic rectangles. But for the presheaf/sheaf view, it is cleaner to treat this as a category:

- Objects: dyadic rectangles R.
- Morphisms: refinement maps R' -> R whenever R' is a refinement piece of R (equivalently, R' ⊆ R with aligned dyadic boundaries).
- Composition: if R'' refines R' and R' refines R, then R'' refines R.

This is again a thin category: between any two rectangles there is at most one refinement morphism.

Calling this structure a “site” is a way of emphasizing that it plays the role of a space with a notion of coverage. In ordinary sheaf theory, open sets cover a space; here dyadic rectangles cover the n×n grid, and each rectangle is covered by its dyadic children. The key difference is that we are working in a finite, discrete, perfectly hierarchical world. That discreteness is not a limitation—it is what makes exact combinatorial constraints possible.

So: the dyadic rectangle poset is not just bookkeeping. It is the multiscale geometry that indexes the constraints.

5.2 “One point per dyadic box” as a compatibility constraint across covers

Now represent a permutation w in S_n as its permutation matrix: the set of n points (i, w(i)) in the n×n grid, exactly one per row and one per column. Consider the function that counts how many points fall in a given dyadic rectangle R:

count_w(R) = number of i with (i, w(i)) in R.

The dyadically well-distributed (DWD) condition can be stated as a family of equalities:

For every dyadic rectangle R of a specified shape (and across all scales), count_w(R) is exactly what uniformity would demand.

In the simplest slogan form used in this project’s narrative: “one point per dyadic box,” meaning that at certain rectangle shapes the count is exactly 1. More generally, depending on the rectangle’s area and the specific definition used, the constraint is an exact count tied to scale. The structural point does not depend on the particular normalization; what matters is that the constraints are exact and scale-indexed.

In a sheaf-like interpretation, the key feature is compatibility across refinement. If a rectangle R is covered by its children R1,…,Rk, then any global point configuration satisfies:

count_w(R) = sum_j count_w(Rj).

This is not an additional constraint; it is a tautology about counting. But when you impose exact constraints at many scales simultaneously, this tautology becomes a powerful consistency filter. You are no longer free to assign counts independently at different scales: fine-scale constraints must sum correctly to coarse-scale constraints. The dyadic system makes this particularly rigid because every cover is a clean binary refinement, repeated across m levels.

So the DWD property is best seen as a demand that the point configuration be *perfectly balanced under every dyadic cover*. In the language of gluing, the counts on refined pieces must

glue to the count on the parent, and the entire hierarchy must remain consistent across all refinement paths.

This is exactly where modularity becomes possible. A hierarchy of constraints can do two things at once:

- enforce global regularity (no drift from scale to scale), and
- isolate local degrees of freedom (in certain blocks, there may be exactly two admissible micro-configurations that both satisfy the constraints).

A naive reader might think “constraints reduce freedom.” The dyadic story is subtler: constraints reduce freedom in most directions but concentrate it into discrete, well-separated switches.

5.3 DWD permutations as global sections under exact local constraints

To complete the presheaf/sheaf analogy, we want a sense in which a permutation is a “global section” of something defined locally on dyadic rectangles.

One workable abstraction is this:

- For each dyadic rectangle R, define the set A(R) of locally admissible point patterns inside R that could appear as the restriction of a permutation matrix, and that satisfy the desired local count constraint on R and its sub-rectangles (to whatever depth is specified).
- If R' refines R, there is a restriction map res_{R,R'} : A(R) -> A(R') that takes a pattern on R and restricts it to the smaller rectangle R'.

This defines a presheaf-like assignment: rectangles map to admissible local data, and refinements map to restrictions. In a strict sheaf setting, you would specify gluing axioms: compatible local pieces glue uniquely to a global piece. Here the world is discrete and the object (a permutation matrix) is global from the start, so one does not usually build it by gluing. But conceptually, the point is the same: a global permutation is exactly a choice of local data on all rectangles that is compatible under restriction and satisfies the global “one per row/column” condition.

Under this view, a DWD permutation is a global section that satisfies extra axioms: it lies in a distinguished subset of the global sections singled out by the dyadic constraints at every scale.

This viewpoint makes two aspects of the hypercube construction feel almost inevitable:

1. **Why there are bits.** If the local admissible set A(R) for certain rectangles has size 2 and these choices are independent across many rectangles, then the space of global sections

over those rectangles is a product of 2's, hence a Boolean lattice. The cube is a space of global sections parameterized by binary switches.

2. **Why the bits commute.** Compatibility under restriction is what prevents interference. If two rectangles are arranged so that their refinements do not overlap in a conflicting way, then choosing one local pattern in one rectangle and another pattern in a different rectangle produces a compatible family. That is the presheaf/sheaf manifestation of commutation: local modifications in independent regions glue in either order.

So the DWD class is not just a subset of permutations; it is the solution set to a multiscale gluing problem with exact constraints. The proof that the interval is a cube is, in this language, a proof that the solution set factors as a large product of 2-element choices.

5.4 Why this viewpoint generalizes: base-b partitions, alternative refinement systems
The main reason to adopt the presheaf/sheaf posture is that it makes clear what is special about dyadic structure and what is not.

What is special about dyadic (base 2) in the original theorem is the exact alignment between:

- the size n = 2^m,
- the binary refinement tree of intervals and rectangles, and
- the existence of local regions where there are exactly two admissible micro-patterns.

But the abstract mechanism does not require base 2. It requires:

- a multiscale refinement category (a hierarchy of partitions),
- exact compatibility constraints across refinements, and
- a regime where many local pieces admit a small discrete set of consistent micro-patterns (ideally two, but more generally small finite sets).

From that perspective, several generalizations become natural.

Base-b partitions. If n = b^m, one can define b-adic intervals and rectangles and impose "uniformity across scales" constraints adapted to base b. In quasi–Monte Carlo theory, digital nets exist in many bases, not only base 2. The key question then becomes: does the local admissible set at certain blocks become b-ary rather than binary? If so, the emergent product region might be a product of b-element chains (or products of small posets), yielding intervals that are not Boolean but still strongly factorized.

Alternative refinement systems. Dyadic rectangles are one particular site. Others include:

- partitions induced by group actions or root systems,
- multiscale decompositions on graphs or lattices,
- hierarchical decompositions of other incidence structures (flags, subspaces, tableaux-like objects).

The presheaf/sheaf lens tells you what to look for: define a refinement category, define local admissibility and restriction, impose exact local constraints that propagate across scales, and then ask whether the global solution space contains a large product region.

New ambient objects and new posets. The DWD condition was phrased using permutation matrices because S_n provides a convenient concrete model. But the "global section under multiscale constraints" idea can be applied to other combinatorial objects that admit a notion of local restriction and global compatibility. If those objects also carry a partial order with meaningful intervals, one can ask the same question: do the constraints carve out a product-shaped interval?

So the presheaf/sheaf view is the bridge from "dyadic uniformity is a special trick that works in S_{2^m}" to "multiscale gluing laws can induce product emergence in many settings." That is the direction this paper wants to take: not away from the original theorem, but through it, using it as a prototype for a broader algebraic and methodological program.

6. HoTT Recasting: Intervals as Types, Cubes as Product Types

Category theory already gives a crisp statement of "cube = product," but homotopy type theory (HoTT) gives something slightly different: it gives a way to state the same phenomenon as an equivalence between types that carries along exactly the structure you care about, no more and no less. In the present setting, HoTT is useful not because Bruhat order secretly has higher homotopical content, but because HoTT provides a disciplined vocabulary for: (i) defining the object of interest as a subtype, (ii) expressing "cube-ness" as a product type, and (iii) distinguishing computational content from proof content.

The Bruhat-hypercube theorem is, at heart, a claim that a certain collection of permutations (those between x and y) is exactly parameterized by d independent bits. In HoTT, "parameterized by bits" is not metaphorical: it is literally the statement that the interval type is equivalent to the product type (Fin 2)^d. The traditional combinatorial proof constructs the parameterization and verifies that the order structure is preserved. HoTT packages that as a structured equivalence: an explicit pair of inverse functions with proofs that they are inverses, plus proofs that they respect the order relation.

This section lays out that packaging. The payoff is conceptual clarity: the “AI found the pattern, humans proved the theorem” workflow becomes, in HoTT terms, “a generator suggested an encoding; humans exhibited an equivalence of types and proved it respects the relevant structure.”

6.1 The interval type: V = { z : S_n | x <= z <= y } as a sigma/subtype

In HoTT, one begins by representing the ambient set S_n as a type. For our purposes, we can treat it as a finite set-level type: a type whose identity types are mere propositions, i.e., an h-set. (This is the HoTT way of saying: equality is ordinary equality; there is no higher path structure we intend to use.)

Fix two permutations x and y in S_n with x <=_B y in Bruhat order. Define the interval type V as a subtype (a sigma-type) consisting of a permutation together with a witness that it lies between x and y:

V := Σ (z : S_n), (x <=_B z) × (z <=_B y).

Here Σ denotes a dependent sum (sigma type), and × denotes product. Informally: an element of V is a triple (z, p, q) where z is a permutation, p is a proof that x <=_B z, and q is a proof that z <=_B y.

Two remarks matter immediately.

First, Bruhat comparability is proposition-valued: for any a,b in S_n, the statement a <=_B b is either true or false, and if true it has no additional data we care about. In HoTT terms, a <=_B b is naturally treated as a mere proposition. That means the proof components p and q carry no computational payload; they are certificates.

Second, V is the right formalization of “the interval [x,y]” because it is the exact type of objects you mean to study, not an informal set with implicit conditions. It also makes “structure preservation” precise: if you later define an order relation on V inherited from Bruhat order, you can ask whether a proposed encoding respects it.

If one wishes to keep the type even closer to the classical “set of elements between x and y,” one can view V as a subset type:

V := { z : S_n | x <=_B z and z <=_B y }.

In many HoTT formalisms, this is implemented via sigma types anyway. The important point is that the interval is a type whose inhabitants are precisely the between-elements, with proofs carried along explicitly.

6.2 The cube type: Cube_d = (Fin 2)^d and its coordinate structure

Next define the cube type. The Boolean lattice B_d is the poset of subsets of {1,…,d}, which is equivalent to the set of bitstrings of length d. In HoTT, "bitstring of length d" is naturally the function type from a finite index set to a two-element type. There are several equivalent presentations. A convenient one is:

Fin 2 : the finite type with two inhabitants (call them 0 and 1).
Fin d : the finite type with d inhabitants (indices 0,…,d−1).
Cube_d := Fin d -> Fin 2.

This is the most explicit "coordinate" version: an element of Cube_d is a function that assigns to each coordinate index i a bit.

If you prefer a tuple notation, you can write:

Cube_d := (Fin 2)^d,

meaning a d-fold product. In HoTT this is literally a product type if d is fixed and small, or equivalently a dependent function type over Fin d for general d. Both capture the same idea: the cube is the type of d independent 0/1 choices.

Now define the cube's order structure, because we ultimately want an order-respecting equivalence. The Boolean lattice order is inclusion, which corresponds to coordinatewise comparison on bits. Let <=_2 be the order on Fin 2 with 0 <= 1. Then define:

For a,b : Cube_d, define a <= b if for all i : Fin d, a(i) <=_2 b(i).

This is a proposition-valued relation (a mere proposition): it says every coordinate of a is <= the corresponding coordinate of b. This is exactly the componentwise order that makes Cube_d into the Boolean lattice B_d.

This makes the "coordinate structure" completely explicit. Projections are simply evaluation at coordinates: given i, the map a |-> a(i) is the i-th coordinate function. Faces and subcubes are defined by fixing some coordinates, varying others.

So if the Bruhat interval is truly a cube, there should exist an encoding map from permutations to bitstrings such that Bruhat comparability becomes coordinatewise inclusion. HoTT is well suited to stating exactly that.

6.3 The theorem as an equivalence of structured types (order-respecting equivalence)

The combinatorial theorem has two layers:

- a bijection between elements of [x,y] and bitstrings of length d, and
- the claim that the order on [x,y] matches the Boolean lattice order under that bijection.

In HoTT, “bijection” is naturally expressed as an equivalence of types:

V ≃ Cube_d.

An equivalence is stronger than a mere pair of functions; it includes proofs that they are inverses up to equality. Concretely, one can present it as:

- f : V -> Cube_d (encoding: sends an interval element to its bitstring)
- g : Cube_d -> V (decoding: constructs the interval element corresponding to a bitstring)
- proofs that g(f(v)) = v for all v : V and f(g(c)) = c for all c : Cube_d.

But because we care about order, we want more than a bare equivalence. We want it to respect the inherited order relations. Let <=_V denote the order relation on V induced by Bruhat order:

For v1 = (z1, _, _) and v2 = (z2, _, _), define v1 <=_V v2 iff z1 <=_B z2.

Then an order-respecting equivalence means:

For all v1, v2 : V,
v1 <=_V v2 iff f(v1) <= f(v2) (in the componentwise cube order).

This is the precise HoTT version of “the interval is isomorphic to a Boolean lattice.” It is not merely that there are 2^d elements; it is that the order structure is exactly the same.

In the original theorem, the encoding f is given by reading off local block choices (bits) from the dyadic construction. The decoding g is given by assembling a permutation matrix from those choices, and the proof that it lies between x and y includes exactly the checks that the assembled object satisfies the dyadic constraints and sits in the intended Bruhat interval. Order preservation is proved by showing that flipping a bit corresponds to a cover move, and that inclusion of bit sets corresponds to reachability in Bruhat order within the interval.

HoTT doesn’t change the mathematics; it changes what you can say cleanly. It lets you state, in one sentence, what the human proof accomplished:

There exists an equivalence V ≃ (Fin 2)^d such that the induced order on V is transported to the coordinatewise order on (Fin 2)^d.

In a sense, the “human step” becomes the act of producing a structured equivalence: encoding + decoding + coherence + order transport.

6.4 Truncation levels and proof relevance: what is set-level vs higher structure

HoTT is often associated with higher homotopical structure, so it is important to be explicit about what we are and are not claiming. In this paper’s use, everything is set-level unless we deliberately add additional structure.

- S_n is treated as a finite set type (an h-set).
- The relations x <=_B y and x <= y in Cube_d are treated as mere propositions.
- The interval type V is therefore also an h-set: its inhabitants are determined by the underlying permutation, because the proof components are proof-irrelevant.

This has two practical consequences.

First, the equivalence V ≃ Cube_d behaves like an ordinary bijection at the level of elements. There is no hidden higher structure lurking in equality; equality is just equality of permutations or bitstrings.

Second, proof relevance becomes a controlled choice rather than an accident. In ordinary mathematics we routinely treat order relations as having no data beyond truth. HoTT makes that explicit: if you choose to treat <= as a mere proposition, then any two proofs of x <= y are equal (or at least indistinguishable for all purposes). This aligns perfectly with the poset-as-thin-category translation: in a thin category, there is at most one morphism between objects, which corresponds to proof irrelevance of the order relation.

So, at the baseline level, HoTT is being used as a clarity tool: it separates the computational content (the encoding of a permutation as a bitstring) from the proof content (the certificates that the permutation lies in the interval and that order is preserved). That separation is philosophically aligned with the AI-assisted discovery narrative:

- The machine suggests an encoding (computational artifact).
- The human supplies the proof that the encoding is correct and structural (proof artifact).

If one wants to go further, HoTT offers optional enrichment: one could consider the nerve of the poset, or cubical structures, or higher coherence conditions in a non-thin setting. But the present paper does not need that. The point is simply that the hypercube interval is most naturally expressed as an equivalence with a product type, and that this makes “independent flips” literally a product decomposition rather than a geometric metaphor.

In summary: the HoTT recast makes the theorem read as an explicit statement of *type-level factorization*. The interval is a subtype carved out by conditions; the cube is a product type of bits; the theorem is an equivalence that transports order. That is exactly the “product emergence” idea in its most precise form: many degrees of freedom become coordinates, and the coordinates become the structure.

7. Cubical and Simplicial Perspectives: Faces, Nerves, and Higher Commutation

Once we have translated “poset” into “thin category” and “cube” into “product,” it becomes natural to ask what the corresponding combinatorial topology looks like. This is not an attempt to turn a discrete theorem into a mystical topological parable. It is a practical move: nerves, simplicial sets, and cubical complexes provide a compact language for describing *coherent families of commutations*. In a cube interval, commutation is not a one-off phenomenon. It occurs in every dimension: any subset of flips defines a face, and all flip orders lead to the same endpoint. That is exactly what cubical and simplicial formalisms are designed to encode.

At the same time, one must be disciplined. Not every topological construction adds insight. Some merely rename facts. The purpose of this section is therefore twofold: (i) to explain what the nerve and the cubical viewpoint are actually saying in this context, and (ii) to draw a bright line between genuine clarification and overinterpretation.

7.1 Nerve of a poset: how intervals become simplicial data

Given any small category C, one can form its nerve N(C), a simplicial set whose k-simplices correspond to composable chains of k morphisms. For a poset P viewed as a thin category, this becomes extremely concrete.

- Objects of P become 0-simplices.
- A comparable pair x <= y becomes a 1-simplex (an edge).
- A chain x0 <= x1 <= ... <= xk becomes a k-simplex.

In other words, the nerve of a poset is a simplicial object that encodes all chains at all lengths. If you apply this construction to an interval [x,y], then the nerve N([x,y]) captures the entire internal chain geometry of that interval, not just its Hasse diagram.

Why is this helpful? Because it packages “all possible ways to move upward” into a single object. In a generic poset, the set of chains can be messy; the nerve records exactly how messy. In a Boolean lattice, the chains have a canonical combinatorial description: every maximal chain corresponds to an ordering of the d coordinate flips, and the number of maximal chains is d! (each permutation of flips gives a distinct maximal chain).

So the statement “the interval is B_d” implies a strong statement about nerves:

N([x,y]) is isomorphic (as a simplicial set) to the nerve of B_d.

You can read this as: every chain in the Bruhat interval is exactly a chain in a cube, and vice versa. There are no extra relations, no missing chains, no accidental collapses. The simplicial data is exactly that of a cube’s face lattice.

This provides a second way to understand “independent flips.” Independence is not only visible at the 1-skeleton level (edges) or even at the 2-skeleton level (commuting squares). It is visible in the entire simplicial stratification: chains correspond to sequences of flips, and the equivalences among these sequences are exactly those forced by product structure.

7.2 Cubical complexes and Boolean lattices: faces as subsets of flips

The simplicial nerve viewpoint emphasizes chains. The cubical viewpoint emphasizes faces. Since the object in question is literally a cube (a Boolean lattice), the cubical vocabulary is especially natural.

A d-dimensional cube has:

- vertices labeled by bitstrings in {0,1}^d,
- edges corresponding to changing one coordinate,
- 2-faces (squares) corresponding to changing two coordinates in either order,
- and in general k-faces corresponding to varying k coordinates while fixing the others.

In the Boolean lattice B_d, a face is determined by choosing:

- a subset F of coordinates allowed to vary, and
- a fixed assignment on the remaining coordinates.

Equivalently, a k-face corresponds to a choice of k flips that are “active” and d−k flips that are held constant. Every face is again a smaller cube: B_k.

Now return to the Bruhat interval [x,y] that is isomorphic to B_d. The “flips” are the local binary choices (block choices) arising from the dyadic constraints. A subset of flips corresponds to restricting to those interval elements where only those flips may vary. That sub-poset is itself an interval, and it is itself a cube of dimension equal to the number of flips in the subset.

So the cubical complex viewpoint reads as:

- Each flip is a coordinate axis.
- Each subset of flips determines a face subinterval.
- The family of all such subintervals has the incidence relations of a cube.

This is a useful perspective because it isolates the exact meaning of “local move.” A single Bruhat cover move corresponds to crossing a 1-face. But the theorem says much more: the interval contains an entire cubical subcomplex in which every higher-dimensional face is present with the expected incidence pattern.

In short: the interval is not merely “cube-like” in its adjacency. It has the entire face structure of a cube, across all dimensions up to d.

7.3 Higher commutation: from squares (2-faces) to k-faces in large cubes

The phrase “commuting squares” is often used as a shorthand for independence. But in a d-cube, the true independence statement is stronger: any finite set of independent flips can be applied in any order and yields the same endpoint. The square is simply the first nontrivial case.

In a product 2^d, choose k distinct coordinates i1,…,ik. Starting from a vertex v, flipping these k coordinates yields a unique vertex v', and the path taken does not matter. There are k! different orders in which one could apply these flips, and they all land at the same v'. The entire set of intermediate vertices forms a k-dimensional face.

Categorically, in a thin category, this is “higher commutation” in the strongest possible sense: there is at most one morphism from the start to the endpoint, so all composites must coincide whenever the intermediate steps exist. What is nontrivial is that every required intermediate step exists and that all the expected vertices appear.

Combinatorially, this is the content of the Boolean lattice property:

- The interval elements correspond to subsets of flips.
- The partial order is inclusion of subsets.
- A chain corresponds to adding flips one by one.
- Any ordering of the same flip set yields a chain with the same endpoints.

So the “higher commutation” structure is not an extra theorem on top of the cube identification; it is the cube identification viewed at full depth. Once you have B_d, you automatically have coherent commutation at every k <= d.

This matters for two reasons.

First, it gives a robust diagnostic for product structure. If you see many squares but they fail to assemble into consistent higher faces, you may have local commutation without global factorization. A true cube interval is characterized by the existence of *all* faces and the correct incidence.

Second, it gives a clean way to restate what the dyadic mechanism accomplished. The dyadic constraints did not merely create many available local moves; they created a regime where those moves are globally compatible under arbitrary combinations. That is exactly what it means for flips to be genuinely independent.

7.4 What topology can and cannot add here: clarity vs overinterpretation

The nerve and cubical perspectives can clarify, but they can also tempt. The temptation is to treat the appearance of a huge cube as a deep topological event, as if the poset were secretly a manifold and the cube were a cell decomposition of some hidden space. That is usually not the right interpretation in this setting.

What topology can add, legitimately, is:

- A compact language for "all chains" (via nerves) and "all faces" (via cubical complexes).
- A way to talk about coherence: not just pairwise commutation, but commutation of any finite family of moves.
- A way to compare substructures: a cube interval is a particular kind of subcomplex, and one can ask for classification of such subcomplexes or for invariants that detect them.

Topology can also help with certain honest combinatorial invariants that are naturally expressed topologically: for instance, the order complex of an interval, shellability properties, Möbius functions, and related combinatorial-topological data. If the interval is B_d, those invariants are computable and may feed into downstream representation-theoretic quantities.

What topology cannot add, at least not without additional structure, is:

- It cannot turn the theorem into a statement about continuous geometry in any direct sense. The cube is a discrete poset, not a geometric cube embedded in Euclidean space in a way that matters intrinsically.
- It cannot manufacture new meaning by analogy alone. Saying "it's cubical" is not a proof of deeper homotopical significance; it is a description of combinatorial coherence.

So the proper stance is disciplined instrumentalism: use the nerve and cubical vocabulary to make the combinatorial structure more explicit and more checkable, but resist the urge to claim that the result is "topological" in a sense it is not.

In the context of this paper's larger agenda, the real value of the cubical/simplicial perspective is methodological. AI-assisted search can be aimed at discovering coherent commutation patterns. A search objective can reward the closure of squares into faces and faces into higher faces. Those are precisely the signatures that a subregion is approaching product structure. Thus topology here is not a metaphysical overlay; it is a practical language for designing and diagnosing "product emergence."

That concludes the cubical and simplicial thread. We have now expressed the same phenomenon in three tightly aligned languages: as a Boolean lattice in a poset, as a product subcategory in a thin category, and as a cubical object whose higher faces encode coherent

commutation. The next step is to separate what is truly dyadic, what is truly Bruhat-specific, and what is universal—so that the abstraction becomes a genuine template rather than a restatement.

8. What Is Truly Dyadic, What Is Truly Bruhat-Specific, What Is Universal

At this point we have told the same story in three languages: combinatorics (a Boolean lattice interval), category theory (a product subcategory), and HoTT/cubical formalisms (an equivalence with a product type and coherent faces). The abstraction is now strong enough that a critical question becomes unavoidable: which parts of the phenomenon are essential, and which parts are artifacts of the chosen setting?

This separation matters for more than philosophical tidiness. It determines whether the result is a spectacular one-off or a prototype for a broader research program. If the cube interval depends delicately on base-2 arithmetic and peculiarities of type A Bruhat order, then generalization will be narrow. If, however, the cube interval is one instance of a repeatable mechanism—multiscale balance forcing modular degrees of freedom—then we can meaningfully search for analogous "product emergence" in other posets, other Coxeter types, and other multiscale partition systems.

This section therefore decomposes the phenomenon into four layers: the dyadic-specific scaffolding, the Bruhat-specific order geometry, a universal pattern that survives both translations, and an explicit proposed abstraction that packages the universal pattern into a reusable object.

8.1 Dyadic-specific: exact self-similar partitions when n = 2^m

What dyadic contributes is not merely "binary digits." It contributes a perfectly aligned, exact multiscale hierarchy.

When n = 2^m, the index set [n] admits an exact dyadic filtration: intervals split cleanly in halves, repeatedly, down to singletons. This gives three concrete advantages that are difficult to replicate in non-dyadic regimes:

1. **Exact self-similarity at every scale.**
   Every dyadic interval decomposes into two dyadic children, and this is compatible across all levels. There are no remainder blocks, no edge effects, no irregular fragments. In multiscale design, exactness matters: approximate self-similarity rarely yields exact global theorems.

2. **Rectangular product structure of the site.**
   Dyadic rectangles are products of dyadic intervals. Refinement is simply "halve x-

interval, halve y-interval," producing a clean, tree-like refinement category. This makes constraints composable and checkable: fine-scale counts must sum to coarse-scale counts in a rigid, predictable way.

3. **A natural notion of scale-uniformity that is exact, not statistical.**
   In many contexts, "uniform distribution" is an asymptotic or probabilistic statement. The dyadic condition is an exact combinatorial rule: each dyadic region has exactly the prescribed count. Exactness is what makes it behave like a sheaf-like gluing law rather than like a convergence theorem.

A subtler dyadic-specific feature is that the dyadic site naturally produces paired complementary blocks at multiple scales. In the construction, those complementary blocks are precisely where local binary choices live. Dyadic partitions do not merely subdivide; they subdivide into pairs. That pairing is the natural birthplace of "two local patterns" and hence of bits.

If one tries to move away from n = 2^m, one immediately encounters friction: partitions no longer refine perfectly; base-2 alignment breaks; and the precise "one per dyadic rectangle" condition becomes either ill-posed or no longer exact. The phenomenon may persist in weakened form, but the cleanest version of it is genuinely dyadic.

So dyadic is doing real work. It is the scaffolding that creates a perfect multiscale constraint environment and a natural family of two-choice modules.

8.2 Bruhat-specific: cover relations and monotonicity of local flips

Dyadic structure alone does not guarantee that the resulting family of permutations forms an interval, much less that it forms a Boolean interval. The second essential ingredient is the specific geometry of Bruhat order—especially its cover relations and the way local moves interact with the length function.

Several Bruhat-specific facts matter here:

1. **There is a graded notion of minimal upward move.**
   Bruhat order comes with a length function, and covers correspond to increasing length by one via allowed operations (in type A, certain transpositions that increase inversion count appropriately). This is what makes "flip = edge" meaningful: a local change can be shown to correspond to a cover step.

2. **Monotonicity can be proved for the chosen local flips.**
   In a Boolean lattice, turning a bit from 0 to 1 moves upward. For the hypercube interval, one must prove that each chosen block flip is monotone in Bruhat order: it moves you upward within the interval and does not accidentally move sideways or downward. This

is not a generic property of arbitrary local modifications of permutations; it depends on how Bruhat order is defined and on the specific chosen local patterns.

3. **Interference is controlled by Bruhat's local combinatorics.**
   The heart of the cube structure is that flips commute. In Bruhat order, many local moves do not commute: doing one can change whether another remains a cover, or it can force a different endpoint. The construction exploits a special alignment between the dyadic modularity and Bruhat's interaction rules so that interference disappears inside the carved-out region.

4. **Intervals are natural units in Bruhat theory for reasons beyond combinatorics.**
   Bruhat intervals are not merely a poset curiosity; they encode geometry (Schubert cells, Richardson varieties) and representation-theoretic data (Kazhdan–Lusztig phenomena). That background is part of why an interval that is a giant cube is immediately consequential: it injects rigid, computable structure into contexts where complicated behavior is typical.

So Bruhat-specific content is also real. The theorem is not merely "a set of permutations parameterized by bits." It is "a set of permutations parameterized by bits such that Bruhat comparability is coordinatewise inclusion." That coordinatewise inclusion is a claim about the order relation, and the order relation is where Bruhat's distinctive geometry lives.

In a different poset on the same underlying set S_n, the same dyadic family might not form an interval at all. Or it might form an interval but not a cube. The cube emerges from the specific match between dyadic modularity and Bruhat monotonicity.

8.3 Universal pattern: multiscale constraints yielding modular degrees of freedom
Now strip away what is dyadic and what is Bruhat-specific. What remains is the universal pattern that motivates this paper's abstraction:

A hierarchical constraint system can enforce global regularity while simultaneously isolating many independent, discrete local degrees of freedom, producing an internal product structure (often a large product of 2's).

This pattern has a recognizable "shape" that does not depend on permutations or Bruhat order:

- You start with a space of global objects (permutations, point configurations, labelings, etc.).
- You impose constraints indexed by a refinement hierarchy (scales, partitions, covers).
- The constraints propagate downward and upward (fine-to-coarse consistency).
- The allowed global objects become highly structured, but not rigid to a single object.

- The remaining freedom is modular: localized switches that can be toggled independently without violating global constraints.

When the modular switches are binary, the solution space begins to look like a Boolean cube. When the modular switches are k-ary, the solution space looks like a product of small finite sets or small posets.

This is the phenomenon we called product emergence. It is familiar in other domains in different clothing:

- In coding theory, constraints produce structured codewords with independent parity choices.
- In tilings and substitutions, global substitution rules still allow local phase choices.
- In low-discrepancy constructions, multiscale balance conditions still permit discrete design parameters.

What the Bruhat interval theorem adds is an unusually clean and extreme instance: the modular degrees of freedom are numerous enough to reach near the counting ceiling, and the ambient order is rich enough that "product emergence" is not trivial.

So the universal moral is not "dyadic is special" or "Bruhat is special." It is: multiscale exactness can be an engine of modularity, and modularity exponentiates.

8.4 A proposed abstraction: "multiscale sites that induce product subcategories"
We can now propose an abstraction that captures the universal pattern in a way that is explicit enough to guide future search and generalization.

The ingredients are:

1. **A refinement category (a multiscale site).**
   A small thin category R whose objects are "regions" at various scales and whose morphisms are refinements. Dyadic rectangles were one example. Base-b rectangles, tree partitions, or other hierarchical covers are others.
2. **A presheaf of local admissible data.**
   An assignment A that to each region r in R assigns a set (or type) A(r) of locally admissible configurations, with restriction maps along refinement morphisms. This encodes what it means to be locally consistent.
3. **A global-object space with an order.**
   A set G of global objects (permutations, labelings, etc.) equipped with a partial order <=. We view (G,<=) as a thin category C(G).

4. **A compatibility map from global objects to local data.**
   A map that takes a global object g in G and produces, for each region r, a local restriction in A(r), compatible under refinement. In sheaf language, this is the "restriction of a global section."

5. **Exact multiscale constraints defining a distinguished family H ⊆ G.**
   A set of axioms expressible in terms of local data on regions (counts, balances, admissibility), such that H is the set of global objects satisfying them.

The abstraction's central question is then:

When does the induced subcategory C(H) (or an interval subcategory inside it) contain a large product subcategory, typically isomorphic to 2^d or to a product of small factors?

This framing packages the phenomenon in a way that is operational:

- It tells you what to vary (the site R and the constraints).
- It tells you what to measure (product rank, number of independent local switches, abundance of commuting faces).
- It tells you what to prove (that the switches are truly independent and monotone in the ambient order, yielding a product subcategory, often an interval).

In the original theorem:

- R is the dyadic rectangle refinement category.
- A(r) records local point-pattern admissibility and counts.
- G is S_n, ordered by Bruhat order.
- H is the class of dyadically well-distributed permutations.
- The product subcategory appears as a Bruhat interval whose elements correspond to choices among many independent local block patterns.

The advantage of naming the abstraction is not that it replaces the original result. It is that it turns the original result into the first instance of a program:

Find multiscale sites and exact constraints that induce large product subcategories inside interesting posets.

Once stated this way, "AI-assisted discovery" becomes less mysterious and more designable. A search system can vary candidate constraints, candidate local modules, and candidate global objects, scoring them by product signatures: commuting squares, higher faces, coordinate

functors, and ultimately explicit equivalences to product types. The mathematics then becomes the human act of validating and compressing the discovered structure into definitions and proofs.

That is the point of this section: the theorem is dyadic in its cleanest form, Bruhat-specific in its order geometry, but universal in its mechanism. By isolating the mechanism as "multiscale sites inducing product subcategories," we turn the story into a blueprint rather than a one-time surprise.

9. Methodology as Algebra: Designing AI Searches for Product Structure

If the Bruhat-hypercube episode teaches a durable methodological lesson, it is not merely "AI can find patterns." It is that the target pattern—large product structure—can itself be formulated algebraically, and therefore can be searched for deliberately rather than stumbled upon accidentally. Once we recast cubes as products, "AI discovered a hypercube" becomes "AI located a region where the poset factorizes into many independent binary coordinates." That is a structural objective, not a numerical curiosity.

This section proposes that the future of AI-assisted mathematics in combinatorial and algebraic settings will look less like random conjecture-generation and more like structure-seeking: designing search objectives that reward factorization signatures (commuting faces, coordinate functors, product decompositions), building program-synthesis pipelines that explicitly search for parametrizations, and interpreting across n (or across types) as a form of naturality constraint. The human role then becomes sharply defined: not "do the proofs" in a generic sense, but compress the discovered structure into minimal generators and portable definitions.

9.1 From maximizing an invariant to maximizing product rank (largest d with (2)^d)

Most computational mathematics searches optimize a scalar invariant: maximize size, minimize discrepancy, maximize girth, find a counterexample with largest diameter, and so on. Those objectives are often useful, but they can be blunt instruments. The Bruhat-hypercube phenomenon suggests a different kind of objective: maximize not a number attached to an object, but a *structural decomposition* inside an object.

Define, informally, the product rank of a poset region (often an interval) as the largest d such that the region contains a full subposet isomorphic to 2^d (and, in the strongest form relevant here, contains such a subposet as an interval or as a product subcategory witnessed by coordinate projections). This is not a standard invariant in the classical literature under that name, but the concept is straightforward:

- Product rank d means there exist d independent flips whose subsets generate a Boolean lattice of size 2^d inside the region.

- Equivalently, it means the region has a cubical skeleton of dimension d with all faces present and correctly glued.

For Bruhat order, one can specialize:

- The interval product rank of Br(S_n) is the maximum d such that there exists x <=_B y with [x,y] ≅ 2^d.

Now compare this objective to typical invariants. The difference is qualitative:

- Maximizing a scalar invariant can be achieved by many "unstructured" examples.
- Maximizing product rank is inherently about *structure*: you must produce a coherent family of commuting moves that closes in every dimension.

This is precisely why product rank is a good AI-search target. It gives the AI something that is both:

- measurable (at least for moderate n, by detecting cube-like subposets or by checking commutation closure), and
- conceptually meaningful (because product rank corresponds to genuine factorization).

It also naturally introduces tiers of difficulty that can guide staged searches:

1. Find many commuting squares (2-faces) in a region.
2. Find coherent families of commuting squares that close into cubes of dimension 3,4,…
3. Find explicit coordinate projections (candidate functors to 2).
4. Find a full cube interval isomorphic to 2^d.
5. Push d toward the information ceiling.

The Bruhat result is, in this staged language, a near-maximal achievement: it reaches the top tier and pushes d close to log2(n!). That suggests the staged pipeline is not theoretical; it is already empirically validated.

So the methodological proposal is simple: replace "search for examples" with "search for factorization." Product rank becomes the score.

9.2 Program synthesis as functor search: generators as morphisms from parameters to objects
Program synthesis changes the nature of search. Instead of searching directly over objects (permutations, poset elements, candidate intervals), it searches over *procedures* that generate objects. That is the crucial step from brute-force enumeration to hypothesis generation.

In categorical terms, a generator is naturally viewed as a morphism from a parameter space to an object space. Concretely:

- Let P be a parameter type (bitstrings, integer seeds, small combinatorial templates).
- Let G be the space of objects you care about (permutations, pairs (x,y), local flip-sets).
- A synthesized program is a map f : P -> G.

If you want product structure, the most valuable programs are not those that output a single good object. They are those that output a **family** of objects with built-in coordinate behavior. That is, you want programs whose parameter space already has product structure (often P = (Fin 2)^d), and whose output reflects that structure in the ambient poset.

This is exactly what happened in the Bruhat episode, interpreted structurally:

- The AI did not merely find one interval. It repeatedly output permutations with a recurring internal modular pattern.
- Humans recognized that the pattern admits a bit-parameterization, and then proved that varying bits corresponds to independent flips forming an interval.

So program synthesis can be framed as **functor search** in at least two senses:

1. In the thin-category sense: you want a monotone map (order-preserving) from a parameter poset (2^d with inclusion) into the Bruhat poset, such that the image is an interval and the map is an embedding. That is a functor between thin categories.
2. In the HoTT sense: you want a function from a product type (Fin 2)^d into the interval type V, together with evidence that it is an equivalence and respects order.

Either way, the synthesized program is not merely producing data; it is proposing a structured morphism. It is saying, implicitly: “here is a coordinate system.” The human task is then to extract the coordinate system explicitly and verify that it has the desired properties.

This reframing suggests concrete design choices for AI searches:

- Choose parameter spaces that are already products (bitstrings), then look for generators that map those parameters into objects while preserving the product signatures (commutation, monotonicity).
- Score programs not just by the quality of their outputs at a fixed parameter, but by whether varying parameters behaves like independent coordinates in the object space.

That is “methodology as algebra”: you are searching for morphisms that exhibit factorization, not merely for isolated points.

9.3 Multi-n generalization pressure as naturality pressure

One of the most important features of the Bruhat discovery workflow, as described, is that the pattern persisted across multiple n, especially along the powers of two. That persistence is not an accident; it is a search design feature.

In standard empirical modeling, training across multiple input sizes encourages generalization. In the present structural setting, scoring across multiple n plays a role that is very close to **naturality** in category theory: it penalizes ad hoc constructions that only work at one size and rewards constructions that commute with the "change of size" structure.

To make this idea precise in spirit, consider that varying n often comes with canonical embeddings or projections:

- In S_n, there are natural ways to view S_n inside S_{n+k} (by fixing extra points), though Bruhat order behavior under such embeddings is subtle.
- In dyadic regimes, refinement from n = 2^m to n = 2^{m+1} has a clean scaling interpretation: one more level of dyadic partitioning.

A pattern that truly belongs to the multiscale structure should behave coherently under these changes. That coherence is exactly what naturality expresses: a family of constructions indexed by n, compatible with the morphisms that relate different n.

In practice, you do not need full categorical machinery to use this idea. You implement it as multi-n scoring:

- A candidate program is evaluated at several n values.
- Its score is high only if it produces coherent outputs across these n's and the outputs exhibit consistent structural signatures (e.g., cube dimension grows as predicted, local modules scale in the expected way, the same digitwise rule explains all cases).

This is naturality pressure in operational form. It forces the generator to be "definition-like," not "fit-like." It is a computational proxy for what mathematicians value: a construction that scales because it is built from a real principle.

In other words, multi-n generalization pressure is not merely a safeguard against overfitting; it is an engine for extracting invariants. It pushes the search toward patterns that can be stated as uniform rules—precisely the kind of rules that later become definitions (like DWD).

So the workflow becomes:

- AI proposes candidate morphisms f_n : P_n -> G_n across several n.
- Only those families that behave compatibly survive scoring.

- Humans then read off the compatibility as a single uniform definition.

That is a remarkably direct bridge from computation to algebra.

9.4 Human extraction as categorical compression: definitions as minimal generators
If program synthesis is functor search, then the human role is not best described as "the proof step" in an undifferentiated sense. It is best described as **categorical compression**: identifying the minimal structural data that generates everything else.

In the Bruhat episode, the AI surfaced outputs that were successful but opaque. The human mathematicians then performed a characteristic act:

- They named the object class (DWD permutations).
- They isolated the minimal rule ("one point per dyadic rectangle" / digitwise uniformity).
- They proved that this minimal rule implies the emergent product structure (the cube interval), and they proved the order-theoretic properties (interval, Boolean lattice).

This is exactly what "definition as minimal generator" means. A good definition is a compression of many examples into a small generative principle. Category theory makes that act feel natural because it constantly asks:

- What is the universal property?
- What is the minimal data from which everything else follows functorially?
- What is the structure-preserving map that explains the phenomenon?

So human extraction can be described as a sequence of compressions:

1. **Example compression:** many AI-generated instances -> one candidate rule.
2. **Rule compression:** a messy rule -> a clean invariant or local constraint system.
3. **Structure compression:** the invariant -> a product decomposition (coordinate maps).
4. **Proof compression:** many checks -> a small set of lemmas that force the whole behavior.
5. **Portability compression:** the proof -> a statement in a language that travels (product subcategories, equivalence with product types, multiscale sites).

In this view, the theorem is not the end; it is the certificate that the compression was correct. The real intellectual product is the definition and the structural map it enables. That is why the most important mathematical artifacts in AI-assisted discovery are likely to be:

- explicit coordinate maps,
- functorial embeddings of parameter cubes into object spaces,
- and minimal local constraints that guarantee global factorization.

So the methodological thesis of this section is tight:

- Search objectives should be structural (maximize product rank).
- Program synthesis should be interpreted as searching for morphisms that exhibit product structure (functor search).
- Multi-n scoring is a proxy for naturality (compatibility across size).
- Human work is categorical compression: extracting minimal generators (definitions) from successful code.

Put differently: the "new workflow of mathematical discovery" is not merely AI plus proof. It is AI-driven proposal of structured morphisms, followed by human extraction of the universal property those morphisms were approximating. That is methodology behaving like algebra.

10. Beyond Type A: Coxeter Groups, Other Orders, Other Product Phenomena

The Bruhat-hypercube theorem lives in type A because permutations are concrete, permutation matrices admit dyadic constraints, and the symmetric group is the best studied finite Coxeter group. But the conceptual payload of the theorem—the emergence of a large product region inside a richly structured order—does not belong to type A alone. Once we translate the result into "product subcategories induced by multiscale constraints," the natural next question is not "can we repeat the same construction verbatim?" but:

Where else should product emergence occur, and what should it look like when the ambient combinatorics changes?

This section surveys that question in a disciplined way. It separates what transfers cleanly from what does not, identifies natural candidate posets beyond Bruhat order on S_n, sketches what the "cube" might generalize into, and proposes a practical roadmap: a blend of constraints, search objectives, and proof patterns that can be carried into new Coxeter types and beyond Coxeter theory entirely.

10.1 Bruhat order in general Coxeter groups: what transfers cleanly

Let (W,S) be a Coxeter system: W is a Coxeter group generated by simple reflections S with relations determined by a Coxeter matrix. Bruhat order is defined for every Coxeter group, not

only for S_n. Many key features of Bruhat order that we exploited in type A transfer essentially unchanged:

1. **A length function and graded structure.**
   Every Coxeter group comes with a length function ℓ : W -> N defined by reduced expressions in simple reflections. Bruhat order is compatible with length: covers increase length by 1. This graded structure is the backbone of interpreting "local flips" as cover moves and "dimension" as a count of independent choices.

2. **An interval-centric viewpoint is natural.**
   Bruhat intervals [x,y] exist in any W and remain the natural unit for local structure. Many invariants and geometric correspondences are interval-local, so the question "which intervals are products?" remains meaningful.

3. **Thin-category translation still applies.**
   Any poset is a thin category. So "interval is a cube" remains "full subcategory is a product," independent of type. The language of commuting squares and higher faces remains available as a diagnostic for product structure.

4. **Reflection-based descriptions exist broadly.**
   Bruhat order admits characterizations in terms of inversion sets and reflections that generalize beyond permutations. This matters because proofs of monotonicity and commutation often become clean when phrased in terms of inversion-set inclusion or reflection sequences rather than in terms of permutation matrices.

However, what does *not* transfer cleanly is the dyadic geometry. The permutation matrix picture is a type A convenience, not a Coxeter-universal representation. In other types, one must replace the "dyadic rectangles on an n×n grid" with some other multiscale site that is native to the group or to a representation of it.

So the transferable core is Bruhat's graded cover structure and the interval logic. The non-transferable scaffolding is the particular dyadic site that lives on permutation matrices.

That observation already suggests the right strategy: keep Bruhat order (or an analog), but find new multiscale constraint systems that carve out modular degrees of freedom in ways appropriate to each type.

10.2 Candidate targets: weak order, Cambrian lattices, other representation-theoretic posets
If the universal phenomenon is "product emergence inside structured posets," there is no reason to restrict attention to Bruhat order. Representation theory and Coxeter combinatorics provide a whole ecosystem of posets that are built from the same generators and reflections but encode different structural aspects. Several are especially natural targets.

1. **Weak order (left/right weak order).**
   Weak order is defined using reduced words and inclusion of inversion sets in a way that is often closer to distributive-lattice behavior than Bruhat order. It is typically "tamer" than Bruhat order in some respects, and product substructures may be easier to characterize or to force. If one is searching for large product regions, weak order can be a proving ground where the relevant commutation/factorization lemmas are simpler.

2. **Cambrian lattices and related quotient posets.**
   Cambrian lattices (in finite types) arise as lattice quotients of weak order controlled by sortable elements. They encode cluster algebra combinatorics and have deep geometric meaning. Because these objects already possess lattice structure, "product emergence" could manifest not only as Boolean subposets but as explicit products of smaller lattices or as intervals isomorphic to distributive lattices associated to posets of join-irreducibles.

3. **Posets from representation theory and geometry.**
   Bruhat order appears in Schubert geometry, but there are other posets:

   - posets of parabolic cosets,
   - Bruhat order on affine Weyl groups and their finite truncations,
   - posets arising from subword complexes, pipe dreams, and related diagrammatics,
   - and various orders associated to crystals, weights, or admissible sets.
     These posets often have natural local moves and natural decompositions. If one can identify multiscale constraints that isolate commuting local moves, product regions may appear there as well.

4. **Posets from quiver and cluster combinatorics.**
   Since the original theorem already "touches" cluster algebra consequences, it is reasonable to look for product phenomena directly in cluster-adjacent structures: exchange graphs, compatibility posets, or orderings on clusters under mutation constraints. Here "flip" is already a literal mutation, and commutation corresponds to commuting mutations in independent directions.

The methodological point is: target posets where (i) local moves are well understood, (ii) rank/length is available, and (iii) there is enough structure that a product interval would have downstream meaning. Those criteria place weak order, Cambrian lattices, and Bruhat-like posets at the top of the list.

10.3 What “hypercube” might become: products of chains, distributive lattices, zonotopal intervals

In type A with dyadic constraints, the emergent product factors are 2-element chains, yielding 2^d. But outside that exact dyadic regime, the “best possible product” might not be Boolean. Three generalizations are especially natural.

1. **Products of chains (grid-like intervals).**
   Instead of each local module having two states, a module might have k+1 states arranged linearly, yielding a chain C_{k+1}. A product of such chains is a rectangular grid poset. This is still a pure product phenomenon, still has commuting coordinate increments, and still has a clean rank structure. In many settings, “multi-choice local patterns” are more realistic than strictly binary patterns.

2. **Distributive lattice intervals.**
   Boolean lattices are the simplest distributive lattices. In more complicated settings, local constraints might force an interval to be a distributive lattice that is not Boolean—equivalently, the lattice of order ideals of some poset J. This would generalize the “independent flips” story to “partially dependent flips,” where some coordinates constrain others but still admit a clean ideal-parameterization.

In category-theoretic terms, this is still a kind of factorization: the interval is controlled by join-irreducible generators with specified dependency relations.

3. **Zonotopal or cube-like but not cube intervals.**
   In geometry, zonotopes are Minkowski sums of line segments; combinatorially they generalize cubes. In posets and oriented matroids, one often sees cube complexes and zonotopal tilings. It is plausible that in Coxeter-theoretic posets, the correct analog of “cube interval” is an interval whose order complex or face structure corresponds to a zonotope-like object—still generated by commuting directions, but with identifications or dependencies that prevent it from being literally 2^d.

These generalizations are not mere indulgences. They are the natural outcomes of relaxing the two ingredients that made the original cube Boolean:

- dyadic pairing (binary choices) and
- perfect decoupling (full independence).

Once modules are k-ary or partially coupled, the emergent product object changes accordingly. The overarching target remains the same: large regions with coordinate behavior that can be described as products (strict or generalized), witnessed by commuting faces and coordinate functors.

So the right generalization of the theorem is not “find cubes everywhere,” but “find large factorized intervals, and classify the factor types.”

10.4 A roadmap for generalization: constraints, search objectives, proof patterns
Generalizing this phenomenon beyond type A requires a coherent plan. The plan should respect the division of labor we identified earlier: computers are good at surfacing patterns in large combinatorial spaces; humans are good at compressing patterns into definitions and proofs. Here is a roadmap that makes that division explicit and repeatable.

Step 1: Choose the ambient poset with meaningful intervals.
Pick W with Bruhat order, or weak order, or a Cambrian lattice, or another representation-theoretic poset where intervals have recognized significance. Require a grading (length/rank) and a notion of local moves (covers).

Step 2: Define a candidate multiscale site native to the setting.
In type A we used dyadic rectangles on a permutation matrix. In other types, candidates could include:

- filtrations by parabolic subgroups,
- hierarchical decompositions of inversion sets,
- base-b or root-system-based digitizations of coordinates in a geometric representation,
- or refinement categories derived from reduced word structure (subword decompositions at multiple scales).

The point is to produce a refinement hierarchy that can host exact local constraints and natural restriction maps.

Step 3: Impose exact local constraints that plausibly create modules.
Design constraints that are strong enough to enforce global regularity but local enough to permit discrete choices. The guiding heuristic is: constraints should make most degrees of freedom disappear while leaving behind many small “switches.”

In practice, this is where AI search can help: it can propose constraint templates and test which templates lead to large product signatures.

Step 4: Optimize a structural objective, not a numeric curiosity.
Use product rank (largest d with a 2^d interval) as the top-level objective when appropriate, but also allow generalized objectives: largest product of chains, largest distributive interval, largest cube-complex dimension, abundance of coherent commuting faces. Measure commutation closure, not just element count.

Step 5: Extract coordinates and prove monotonicity.
Once a promising family is found, the human proof pattern is almost always the same:

- Identify the local degrees of freedom (coordinates).
- Prove each coordinate move is monotone (upward) in the ambient order.
- Prove independence/commutation (faces close).
- Prove closure under all subsets (full product structure).
- Prove the set is actually an interval (has clear endpoints and contains everything between).

This is the "interval identification plus cube identification" architecture seen in the original theorem, now expressed as a general proof template.

Step 6: Compress into a definition that travels.
Finally, extract a minimal, native characterization of the discovered objects (analogous to DWD), ideally stated in the language of the chosen Coxeter type or poset. This is where the work becomes reusable: it turns one discovered family into a class with a definition, with lemmas, and with the possibility of further theorems.

In short, the path beyond type A is not to carry permutation matrices with us. It is to carry the *mechanism*: multiscale constraints that isolate modular degrees of freedom, producing large factorized intervals detectable by product signatures. With that mechanism framed as an algebraic search objective, and with proof patterns articulated in advance, generalization becomes less like wandering and more like a program.

11. Interpretive Boundaries: Responsible Abstraction Without Mythology

A successful abstraction has a distinctive feel: it makes a phenomenon smaller without making it weaker. It compresses many examples into a short definition, many checks into a short proof architecture, and many ad hoc observations into a portable template. But abstraction can also misfire. It can become decoration, a way of borrowing prestige from a formalism without adding explanatory power. It can become mythology: a way of turning an honest theorem into a story about "new kinds of intelligence" or "deep truths about reality" that the theorem does not actually support.

This section is therefore a boundary-setting exercise. We have translated the Bruhat-hypercube theorem into categorical language, a presheaf/sheaf lens, HoTT equivalences, and cubical/simplicial coherence. Those translations are useful—if we stay honest about what they do and do not claim. The aim here is to keep the paper's abstraction disciplined: to use category

theory and HoTT as instruments of clarity and portability, not as rhetorical amplifiers. And to state concrete norms for what counts as a credible AI-to-proof pipeline, so that the headline story does not outrun the mathematics.

11.1 Abstraction as compression, not as authority

The central ethical posture in this paper is simple: abstraction is a tool for **compression**, not a source of authority.

Compression means:

- The same content can be stated with fewer moving parts.
- The same proof can be organized around a smaller set of lemmas.
- The same phenomenon can be recognized in new settings because the essential pattern has been isolated.

Authority, by contrast, is the social effect where formal language is used to silence questions: "it's categorical, therefore it must be deep," or "it's HoTT, therefore it's foundational," or "AI found it, therefore it's beyond ordinary skepticism." Those moves are not arguments; they are status transfers. They violate the intellectual discipline that mathematics requires.

In the present case, the compression value of abstraction is real and specific:

- "Cube interval" compresses into "product subcategory," which makes the structure portable beyond posets to any thin-category context.
- "Dyadic uniformity" compresses into "gluing law on a multiscale site," which makes it conceivable to design analogous constraints in new settings.
- "Parameterization by bits" compresses into "equivalence with a product type," which isolates the computational content from proof certificates.

None of these compressions creates new truth. They reorganize existing truth. The theorem remains the theorem: a particular interval is a Boolean lattice of large dimension. Abstraction does not elevate it into an oracle about reality; it makes it easier to move, reuse, and generalize.

So the boundary is: we may use abstraction to *reduce* cognitive burden and increase portability, but we do not use abstraction to claim extra epistemic weight.

11.2 Avoiding category-theory decoration: what this translation genuinely explains

Category theory is especially prone to decorative misuse because it has a reputation for depth and generality. The responsible question is therefore: what did the categorical translation *actually* explain here?

It genuinely explained three things.

1. **Independence becomes a structural claim.**
   In a poset story, “independent flips” can sound psychological or heuristic. In the thin-category story, independence is the existence of product coordinates and the closure of commuting faces. The categorical language makes “independence” checkable: do we have the product structure or not?

2. **The cube is re-identified as a universal product, not a special shape.**
   Saying “B_d ≅ 2^d” is not a flourish. It is the precise identification that turns a geometric metaphor into an algebraic decomposition. It also makes it clear what alternative outcomes might look like (products of chains, distributive lattices, etc.).

3. **The AI pipeline can be formulated as searching for morphisms.**
   Program synthesis naturally produces generators, i.e., maps from parameters to objects. Category theory gives a clean way to interpret that: you are searching for a structured map that exhibits a product (an embedding of 2^d, or a functor from a parameter cube). That reframing helps design search objectives and interpret results without narrative inflation.

What category theory did not do is provide a new proof, or reveal hidden higher structures, or somehow “explain why” mathematics exists. It clarified structure and provided a portable template. That is enough; it is also the only legitimate claim.

A helpful self-test is: if we remove the categorical language, do we lose any mathematical content? In this paper, the answer is: we lose portability and design guidance, but not the theorem. That is the correct role for category theory here. If the theorem depended on category theory for its truth, we would be doing a different kind of work.

So the boundary is: category theory is being used as a minimal translation that exposes product structure and guides generalization; it is not being used as an aura.

11.3 Avoiding HoTT inflation: equivalence is not metaphysics

HoTT is even more susceptible than category theory to metaphysical inflation, because it talks about “types,” “paths,” and “equivalences,” language that can easily be misheard as a philosophy of reality. The responsible stance is to be explicit: in this paper, HoTT is used as a disciplined syntax for ordinary set-level mathematics.

When we define the interval type:

V := Σ(z : S_n), (x <= z) × (z <= y),

we are not claiming that Bruhat intervals are homotopy spaces in a deep sense. We are stating, in a proof-relevant language, that an interval element is a permutation together with witnesses that it lies between x and y. The proof components are propositions, not data with higher structure. We are living at set truncation.

When we define the cube type:

Cube_d := (Fin 2)^d,

we are not claiming the cube is a physical object. We are stating that the cube is a product type of independent bit choices.

And when we say the theorem is an equivalence:

V ≃ Cube_d,

we are not claiming that "everything is equivalent" or that equivalence is ontological sameness. We are saying: there is an explicit encoding and decoding that are inverse, and the order structure is transported correctly.

That last point matters because HoTT equivalence can be rhetorically overinterpreted. In HoTT, "equivalence" is the right notion of sameness for types, but it is still a mathematical relation defined by explicit functions and coherence proofs. It does not magically confer meaning beyond the structure specified.

In fact, one of HoTT's virtues is exactly the opposite of mysticism: it separates computation from proof. It makes you say what is computational (the encoding/decoding functions) and what is propositional (the certificates they are inverses, the order-respecting proofs). That is a prosaic and valuable discipline—especially in AI-assisted contexts where computation can be abundant and proof must be explicit.

So the boundary is: HoTT provides a clean formal packaging for "bit-parameterization with structure transport." It does not provide a metaphysical theory of mathematics, and we do not need it to.

11.4 Reproducibility norms: what counts as a successful AI-to-proof pipeline

The final boundary is methodological. The headline "AI found the pattern; humans proved the theorem" can easily become a genre of story that is repeated without standards: the AI says something interesting, humans retrofit meaning, and the narrative claims success. To avoid that,

we need reproducibility norms—criteria that distinguish a genuine pipeline from a publicity loop.

A successful AI-to-proof pipeline in this domain should satisfy at least the following norms.

1. **Artifact transparency.**
   The AI output that seeded the discovery should be preserved in a form that can be inspected: programs, parameters, generated examples, and scoring criteria. If the claim is "an evolutionary system found it," then the evolutionary objective and selection mechanism are part of the mathematical record.

2. **Structural invariants, not anecdotal examples.**
   The pipeline must identify measurable structural signals: cube dimension, commuting-face closure, coordinate projections, interval endpoints, monotonicity checks. A few small-n examples are not enough. The AI's contribution must be presented as evidence of a stable phenomenon, not as a curiosity.

3. **Generalization tests as a gate, not as a bonus.**
   Multi-n evaluation should be treated as essential. A pattern that appears only at one n is not a discovery of structure; it is a discovery of a special case. In the Bruhat-dyadic setting, powers-of-two scaling is part of the claim, so the pipeline must demonstrate persistence across a nontrivial range of m.

4. **Human-readable compression.**
   The end product must include a clean definition that generates the phenomenon (e.g., DWD), plus a proof architecture that a human mathematician can follow without the AI system. If the result cannot be stated and proved without referencing the AI's internal behavior, it has not been properly converted into mathematics.

5. **A clear separation of roles.**
   The record should distinguish:

   - what the AI produced (candidate generators, patterns, code),
   - what humans inferred (definitions, conjectures), and
   - what humans proved (lemmas, equivalences, order preservation).
     This is not about credit allocation; it is about epistemic clarity. Confusing these layers is how mythology forms.

6. **Failure modes documented.**
   A serious pipeline records not only the successes but the near-misses: patterns that worked at small n but failed at larger n; objectives that produced cube-like adjacency but

not true product structure; constructions that were not intervals. This matters because it calibrates how fragile the phenomenon is and prevents overgeneralization.

When these norms are satisfied, the narrative becomes legitimate: AI-assisted search has become a reproducible instrument for discovering product structure, and the conversion to theorem has been made explicit.

This is the proper ending of the abstraction arc. We have translated the theorem into category theory and HoTT not to inflate it, but to compress it; not to assert new authority, but to expose structure; not to create mythology, but to define a repeatable, checkable workflow. With these interpretive boundaries in place, "product emergence" can become a research program rather than a headline genre.

12. Conclusion: The Reusable Claim

This paper began with a very specific event: an AI-guided search surfaced an unexpected family of Bruhat intervals in S_n that are enormous hypercubes, and humans extracted and proved the general theorem. That story is interesting in its own right, but the point of recasting it in abstract algebra, category theory, and HoTT was never to make it sound grander. It was to make it reusable.

Reusability is the right test of value here. A one-off construction can be impressive; a reusable mechanism can become a program. What we have tried to do is isolate a minimal set of claims—mathematical, structural, and methodological—that remain true when you remove the surface details (permutation matrices, dyadic rectangles, type A) and keep the core phenomenon: multiscale constraints can carve out high-dimensional product regions inside complicated ordered worlds, and AI can be engineered to find those regions.

The conclusion therefore does not "wrap up" so much as it states what survives translation: what one can carry forward as a clean conceptual package.

12.1 The mathematical claim: large Boolean intervals are large product subcategories

At the level of pure mathematics, the core claim can be said without mentioning AI at all.

A Boolean lattice B_d is the archetype of a product: it is the d-fold product of the 2-element chain. When an interval [x,y] in a poset P is isomorphic to B_d, this is not merely an isomorphism of sets, nor merely a coincidence in the Hasse diagram. It is an identification of structure: the interval is literally a product object inside the thin category associated to P.

So the reusable mathematical claim is:

- A “hypercube interval” is exactly a full subcategory that factorizes as a product of d rank-1 factors.
- Its 1-skeleton, 2-faces, and all higher faces are forced by that factorization: independent coordinates yield coherent commutation at every dimension.
- The dimension d is not an aesthetic number; it is a measure of how many independent coordinate directions the ambient poset admits *in one coherent region*.

In the original Bruhat setting, the near-maximal growth $d \asymp n \log n$ matters because it shows that the ambient poset, despite its complexity, contains regions whose internal logic is almost maximally factorized. That is a strong existence statement about the internal geometry of Bruhat order.

But the point of phrasing it as “product subcategory” is that it becomes portable. In any poset you care about, you can ask:

- What is the largest d for which there exists an interval isomorphic to 2^d?
- More generally, what is the largest product of small factors that occurs as an interval?

That is a clean mathematical question independent of the original story.

12.2 The structural claim: dyadic multiscale uniformity is a gluing principle that yields products

The second reusable claim is not about Bruhat order specifically. It is about a mechanism for producing product structure: gluing under multiscale constraints.

Dyadic uniformity is best understood not as a trick, but as an exact compatibility law indexed by a refinement hierarchy. The dyadic rectangle site provides:

- a multiscale indexing category (regions and refinements),
- a notion of restriction (how a global permutation matrix looks on a region), and
- exact local constraints (uniform counts at each scale) that must glue consistently across covers.

The remarkable part is that this gluing principle does two things simultaneously:

- It enforces global regularity so strongly that the solution set becomes rigid enough to be describable in a clean language (DWD).

- It leaves behind many discrete “switches”—local two-pattern modules—that do not interfere, producing a high-dimensional product family.

This is the structural core of product emergence:

- A sufficiently coherent multiscale constraint system can convert global complexity into local modularity.
- Modular degrees of freedom exponentiate into large product regions.
- Independence is not a heuristic; it is witnessed by face closure and coordinate projections.

Dyadic partitions are the cleanest case because the refinement system is exact and self-similar. But the gluing perspective immediately suggests generalizations: base-b partitions, other refinement sites, other local constraint families. In those generalized settings, the “cube” may become a product of chains, a distributive lattice, or a more zonotopal factorization object. The dyadic result is therefore a prototype: it shows how gluing can yield factorization at scale.

12.3 The methodological claim: search can be engineered to detect latent product structure
The third reusable claim is methodological: if the target phenomenon is product structure, then the search can be designed to aim directly at product signatures. The key move is to stop treating pattern discovery as a mystical act and treat it as a **structural optimization problem**.

The paper’s methodological translation proposes:

- Replace scalar objectives (“find something large”) with structural objectives (“maximize product rank,” “maximize coherent face dimension,” “find coordinate projections”).
- Treat program synthesis as the search for parametrizations: maps from a product parameter space into an object space, ideally monotone and structure-preserving.
- Use evaluation across multiple n (or multiple types) as naturality pressure: only patterns that scale survive.

Under these norms, “AI found the pattern” means something precise:

- The AI found a generator whose outputs exhibit robust product signatures across sizes and instances.
- Humans extracted the minimal definition that explains those signatures.
- Humans proved that the extracted definition induces a genuine product interval, not merely a cube-like adjacency graph.

This is not a promise that every search will succeed. It is a claim that the workflow can be made principled: product structure leaves detectable traces (commuting squares closing into higher faces, coordinate functors, interval endpoints), and those traces can be operationalized into objectives.

So the methodological claim is that AI can become, in this niche, a controllable instrument: a device for proposing coordinate systems in combinatorial universes where human intuition is systematically weak.

12.4 Open problems: classification, alternative scales, other Coxeter types, new objectives
Finally, reusability demands that we end not with applause but with questions that can be carried forward. The following open problems are natural continuations of the program suggested by this paper.

1. **Classification of product intervals in Bruhat order.**
   Even within type A, one can ask for a structural classification: which Bruhat intervals are Boolean lattices, which are products of chains, and what invariants detect them? The dyadic family provides near-maximal examples, but it almost certainly does not exhaust the landscape of factorized intervals.

2. **Alternative scale systems in type A.**
   Dyadic refinement is canonical for n = 2^m. What happens for n = b^m (base-b sites) or for n with mixed prime factors? Are there exact multiscale constraints that yield large product-like intervals in non-dyadic sizes, perhaps with factors of varying arity? What is the right "uniformity across scales" notion that remains exact rather than asymptotic?

3. **Other Coxeter types and their native sites.**
   In types B, D, and beyond, Bruhat order and weak order remain available, but the natural combinatorial representations differ. What are the right multiscale sites there—parabolic filtrations, inversion-set hierarchies, geometric representation partitions—and do they induce product intervals of comparable extremal size? Is there an analog of "digitwise well-distributed" that is intrinsic to other types?

4. **Other posets in representation theory.**
   Weak order, Cambrian lattices, and cluster-related posets are natural arenas for product emergence. What forms of factorization occur there? Do we see large distributive intervals? Do we see zonotopal structures rather than Boolean ones? Which structures have meaningful downstream consequences (bounds, geometric decompositions, polynomial positivity phenomena)?

5. **New search objectives and diagnostics.**
   Product rank is only one objective. Others include: maximize face-closure depth;

maximize the number of coordinate functors to small chains; maximize the size of a distributive interval; find families whose order complexes have particular shellability or tiling properties. Developing a library of such objectives would turn "AI-assisted structural discovery" into a repeatable craft.

6. **A principled notion of "multiscale site induces product."**
   The abstraction proposed here—multiscale sites with gluing constraints that yield product subcategories—should itself be formalized: what conditions on a site and constraint system guarantee the existence of many independent modules? Can one prove general theorems that predict product dimension from the refinement structure and local admissibility counts?

These questions are the real conclusion. The Bruhat-hypercube theorem is a milestone, but its deeper value lies in how it suggests a new style of work: choose a structured poset, define a multiscale constraint system, search for modular degrees of freedom, and then compress the result into a clean algebraic definition and proof.

That is the reusable claim: product structure is not only something one can prove after it is seen; it is something one can actively look for—with definitions, objectives, and proof templates designed in advance.

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The author has published over 4,800 AI-assisted papers at: https://www.researchgate.net/profile/Douglas-Youvan/research . Google Scholar has indexed these preprints, and if you want a particular reference, the AI mode of a Google search (Gemini) has efficiently catalogued these preprints.

$$n = 2^m, \qquad d \;=\; \frac{n}{2}\,\log_2(n) \;=\; 2^{m-1}m, \qquad d \;\le\; \log_2\!\big(|S_n|\big) = \log_2(n!)$$

$$[x,y]_B \;:=\; \{\, z \in S_n \mid x \le_B z \le_B y \,\}$$

$$[x,y]_B \;\cong\; B_d \;\cong\; \{0,1\}^d \;\cong\; (\mathrm{Fin}\,2)^d$$

$$V \;:=\; \{\, z : S_n \mid x \le_B z \le_B y \,\} \;\simeq\; (\mathrm{Fin}\,2)^d$$

$$u \le v \iff \forall i \in \{1,\dots,d\},\; u_i \le v_i$$

$$\phi : \{0,1\}^d \to [x,y]_B, \qquad \psi : [x,y]_B \to \{0,1\}^d, \qquad \psi \circ \phi = \mathrm{id},\;\; \phi \circ \psi = \mathrm{id}$$